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Iwasawa Factorization of the Pseudo-Orthogonal Group $SO(5, 1)$ and its Corresponding Lie Algebra $so(5, 1)$

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We carry out a structural analysis of the Lie group $SO(5, 1)$ by obtaining its Iwasawa factorization. The Iwasawa factorization at the Lie algebra level is also obtained.

1. INTRODUCTION

The Lie group $SO(5, 1)$ has been involved in cosmological models. Here we give its structural properties, as well as those of its associated Lie algebra $so(5, 1)$.

The paper is organized as follows: In Section 2, we calculate the maximal ideal or the maximal invariant subgroup of $SO(5, 1)$ and $so(5, 1)$, the nilpotent Abelian subalgebra and subgroup, the Abelian subgroup and subalgebra, the eigenvalue spectrum, and the eigenfunctions of $so(5, 1)$ as well as its hyperplanes, and the Weyl chambers. In Section 3, the Iwasawa decompositions of the Lie group $SO(5, 1)$ and the algebra $so(5, 1)$ are obtained.

2. THE NILPOINT ABELIAN, THE ABELIAN, AND THE MAXIMAL INVARIANT SUBGROUP OF $SO(5, 1)$ AND $so(5, 1)$

To the general pseudo-orthogonal Lie group $SO(n-s, s)$ we can introduce the corresponding Lie algebra (Gourdin, 1967) as

$$[Z_{ij}, Z_{kl}] = g_{jk}Z_{il} - g_{ik}Z_{jl} + g_{il}Z_{jk} - g_{jl}Z_{ik} \tag{1}$$

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with $Z_{ij} = -Z_{ji}$. We specialize to the case of the Lie group $SO(5, 1)$ and the Lie algebra $so(5, 1)$ by putting $g_{ij} = g_i\delta_{ij}$, so that

$$g_1 = g_2 = g_3 = g_4 = g_5 = -g_6 = 1$$

The infinitesmal generators of $SO(5, 1)$ may be chosen as

$$Z_{12}, Z_{13}, Z_{14}, Z_{15}, Z_{23}, Z_{24}, Z_{25}, Z_{26}, Z_{34}, Z_{35}, Z_{45}, Z_{46}, Z_{56} \tag{2}$$

which are reordered respectively as

$$\begin{aligned} Z_{1i} &= X_{i-1}, \qquad & i &= 2, 3, 4, 5, 6 \\ Z_{2i} &= X_{3+i}, & i &= 3, 4, 5, 6 \\ Z_{3i} &= X_{6+i}, & i &= 4, 5, 6 \\ Z_{4i} &= X_{8+i}, & i &= 5, 6 \\ Z_{56} &= X_{15} \end{aligned} \tag{3}$$

Writing explicitly (3) gives

$$\begin{array}{llll} Z_{12} = X_1, & Z_{13} = X_2, & Z_{14} = X_3, & Z_{15} = X_4 \\ Z_{16} = X_5, & Z_{23} = X_6, & Z_{24} = X_7, & Z_{25} = X_8 \\ Z_{26} = X_9, & Z_{34} = X_{10}, & Z_{35} = X_{11}, & Z_{36} = X_{12} \\ Z_{45} = X_{13}, & Z_{46} = X_{14}, & Z_{56} = X_{15} & \end{array} \tag{4}$$

These generators close on the Lie algebra $so(5, 1)$ defined by the commutation relations of Table I, in which $[X_i, X_j] = C_{ijk}X_k$, with $C_{ij} = \mp k, 0$ for $[X_i, X_j] = \pm X_k$, and the corresponding adjoint matrices for the generators can be read off from Table I.

2.1. The Killing Forms of the Generators of *so*(5, 1)

The structure constants of the Lie algebra can be deduced from equations (2.5). From these constants, one constructs the adjoint matrix operators associated with the generators.

The Killing form associated with the generators X_i and X_j of $so(5, 1)$ is

$$B(X_i, X_j) = \text{Trace}(\text{Ad } X_i \text{ Ad } X_j) \tag{5}$$

where

$$X_i, X_j \in so(5, 1), \qquad \text{Ad } X_i = (C_i)_{jk} \tag{6}$$

$(i, j, k = 1, 2, \ldots, n$; $n = 15$, and C_{ijk} are the structure constants. The generators of $so(5, 1)$ can be separated into two sets, depending on whether

Table 1

$i \backslash j$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}
X_1	0	−6	−7	−8	−9	+2	+3	+4	+5	0	0	0	0	0	0
X_2	+6	0	−10	−11	−12	−1	0	0	0	+3	+4	+5	0	0	0
X_3	+7	+10	0	−13	−14	0	−1	0	0	−2	0	0	+4	+5	0
X_4	+8	+11	+13	0	−15	0	0	−1	0	0	−2	0	−3	0	+5
X_5	+9	+12	+14	+15	0	0	0	0	+1	0	0	+2	0	+3	+4
X_6	−2	+1	0	0	0	0	−10	−11	−12	+7	+8	+9	0	0	0
X_7	−3	0	+1	0	0	+10	+13	−13	−14	−6	0	0	+8	+9	0
X_8	−4	0	0	+1	0	+11	+14	0	−15	0	−6	0	−7	0	+9
X_9	−5	0	0	0	−1	+12	+6	+15	0	0	0	+6	0	+7	+8
X_{10}	0	−3	+2	0	0	−7	−0	0	0	0	−13	−14	+11	+12	0
X_{11}	0	−4	0	+2	0	−8	0	+6	0	+13	0	−15	−10	0	+12
X_{12}	0	−5	0	0	−2	−9	−8	0	−6	+14	+15	0	0	+10	+11
X_{13}	0	0	−4	+3	0	0	0	+7	0	−11	+10	0	0	−15	+14
X_{14}	0	0	−5	0	−3	0	−9	0	−7	−12	0	−10	+15	0	+13
X_{15}	0	0	0	−5	−4	0	0	−9	−8	0	−12	−11	−14	−13	0

they have negative-definite or positive-definite Killing forms (Hermann, 1966; Helgason, 1962; Strom, 1971; Pontryagin, 1969; Nagel, 1969). Thus,

$$B(X_i, X_i) = -8, \qquad i = 1, 2, 3, 4, 6, 7, 8, 10, 11, 13 \tag{7}$$

$$B(X_i, X_i) = +8, \qquad i = 5, 9, 12, 14, 15 \tag{8}$$

We denote by L_K and P the sets of generators with negative- and positive-definite Killing forms, respectively:

$$L_K = \{X_1, X_2, X_3, X_4, X_6, X_7, X_8, X_{10}, X_{11}, X_{13}\} \tag{9}$$

$$P = \{X_5, X_9, X_{12}, X_{14}, X_{15}\} \tag{10}$$

The elements of P close on a nilpotent subalgebra of $so(5, 1)$, while those of L_K form the maximal compact subalgebra of $so(5, 1)$ or the maximal ideal of $so(5, 1)$ (Pontryagin, 1969). The compact subalgebra L_K is isomorphic to $so(5, R)$. The commutation relations defining L_K are given in Table II. The generators of $so(5, R)$ are given by

$$Z_{pr} = -\left(X_p \frac{2}{\delta X_r} - X_r \frac{2}{\delta X_p}\right) \tag{10}$$

$(p, r = 1, 2, 3, 4;\ r > p)$. These are explictly realized as

$$\begin{aligned} &Z_{12} \equiv X_1, \quad Z_{13} \equiv X_2, \quad Z_{14} \equiv X_3, \quad Z_{15} \equiv X_4, \quad Z_{23} \equiv X_6, \\ &Z_{24} \equiv X_7, \quad X_{25} = X_8, \quad Z_{34} = X_{10}, \quad Z_{36} \equiv X_{11}, \quad Z_{45} = X_{13} \end{aligned} \tag{11}$$

These close on the Lie algebra $so(5, R)$. Therefore L_K is isomorphic to $so(5, R)$.

Table II. Commutation Relations and the Structure Constants for the Maximal Compact Subalgebra $so(5, R) \equiv L_K$ of $so(5, 1)$

i \ j	X_1	X_2	X_3	X_4	X_6	X_7	X_8	X_{10}	X_{11}	X_{13}
X_1	0	−6	−7	−8	+2	+3	+4	0	0	0
X_2	+6	0	−10	−11	−1	0	0	+3	+4	0
X_3	+7	+10	0	−13	0	−1	0	−2	0	+4
X_4	+8	+11	+13	0	0	0	−1	0	−2	−3
X_6	−12	+1	0	0	0	−10	−11	+7	+2	0
X_7	−3	0	+1	0	+10	0	−13	−6	0	+8
X_8	−4	0	0	+1	+11	+12	0	0	−6	−7
X_{10}	0	−3	+2	0	−7	+6	0	0	−13	+14
X_{11}	0	−4	0	+2	−2	0	+6	+13	0	−10
X_{13}	0	0	−4	+3	0	−8	+7	−14	+10	0

Table III

$i \backslash j$	X_5	X_9	X_{12}	X_{14}	X_{15}
X_5	0	+1	+2	+3	+4
X_9	−1	0	+6	+7	+8
X_{12}	−2	−6	0	+10	+11
X_{14}	−3	−7	−10	0	+13
X_{15}	−4	−8	−11	−13	0

One checks that between the elements of P and L_K the following hold:

$$[L_K, P] \subset P, \qquad [P, P] \subset L_K \tag{12}$$

$$[L_K, L_K] \subset L_K, \qquad B(P, L_K) = B(L_K, P) = 0 \tag{13}$$

Relations (12) show that the subspace P is orthogonal to the subspace L_K with respect to the Killing form. Algebraically, the result implies that one can write

$$\begin{aligned} so(5,1) &= L_K + P \\ &= so(5, R) + P \end{aligned} \tag{14}$$

giving us the first-stage decomposition of $so(5,1)$.

We can now determine the structure of the subset P of $so(5,1)$. The elements of P that mutually commute form the maximal Abelian subalgebra of P. We denote this subset of P by L_A, so that $L_A \subset P$. We define L_A

$$L_A: \quad [X, Y] = 0, \qquad X, Y \in L_A, \quad L_A \subset P \tag{15}$$

The commutation relations of the elements of P are given in Table III. We find that no two generators commute, so that we can arbitrarily choose

$$L_A = X_5 \tag{16}$$

In the general case, one denotes the number of the elements of L_A by l, so that $L_A = \{H_i\}$, $i = 1, 2, 3, \ldots, l$, where l is the dimension of the Lie vector-space of L_A.

2.2. Eigenvalues and Roots of $so(5,1)$

We set up eigenvalue equations for L_A as follows: For the set of all elements X belonging to the original Lie algebra L $so(5,1)$, we have

$$[H, X] = \alpha(H)X \tag{17}$$

where for a given H, $\alpha(H)$ is a real number, known as the eigenvalue of

H, and the element X is the eigenfunction of H. In general, there should be as many different eigenvalues as there are distinct eigenfunctions. However, some eigenvalues may be degenerate, whereby we have multiplicities. The number of different eigenfunctions X having the same eigenvalue $\alpha(H)$ for a fixed $H \in L_A$ is called the multiplicity or the degeneracy of $\alpha(H)$. We denote the set of all such degenerate eigenfunctions by L^α. These quantities have to be computed in order to gain more information about the structure of the Lie algebra $so(5, 1)$. We first find the set of L^α quantities for the case where any eigenfunctions X have an eigenvalue $\alpha(H) = 0$. We denote the set of such eigenfunctions X by L^0, corresponding to $\alpha(H) = 0$, and write

$$L^0 = \{X, [H, X] = \alpha(H)X = 0, \quad X \in so(5, 1)\} \tag{18}$$

From the commutation relations for $so(5, 1)$, we have, for $X_5 \equiv H \in L_A$,

$$\begin{aligned} &[X_5, X_6] = 0, \qquad [X_5, X_7] = 0, \qquad [X_5, X_8] = 0 \\ &[X_5, X_{10}] = 0, \qquad [X_5, X_{11}] = 0, \qquad [X_5, X_{13}] = 0 \end{aligned} \tag{19}$$

Equations (19) imply that

$$L^0 = \{X_5, X_6, X_7, X_8, X_{10}, X_{11}, X_{13}\} \tag{20}$$

The centralizer of L_A in $so(5, 1)$ is L^0, and it is the maximal Abelian or Cartan subalgebra of $so(5, 1)$. The remaining nonzero eigenvalues are called roots, and these are determined from the commutation relations of $so(5, 1)$. We extract L^0 from $L = so(5, 1)$, to have

$$\{X_1, X_2, X_3, X_4, X_9, X_{12}, X_{14}, X_{15}\}$$

The commutation relations of these with X_5 give

$$\begin{aligned} &[X_5, X_1] = X_9, \qquad [X_5, X_2] = X_{12}, \qquad [X_5, X_3] = X_{14} \\ &[X_5, X_4] = X_{15}, \qquad [X_5, X_9] = X_1, \qquad [X_5, X_{12}] = X_2 \\ &[X_5, X_{14}] = X_3, \qquad [X_5, X_{15}] = X_4 \end{aligned} \tag{21}$$

None of these is of the format of equation (17). Therefore X_1, X_2, X_3, X_4, X_9, X_{12}, X_{14}, X_{15} are not eigenfunctions of X_5. We can try to form suitable linear combinations that will be eigenfunctions of X_5. Let

$$\begin{aligned} X_1 + X_9 = Q^+; &\qquad X_1 - X_9 = P^- \\ X_2 + X_{12} = Q^+; &\qquad X_2 - X_{12} = Q^- \\ X_3 + X_{14} = R^+; &\qquad X_3 - X_{14} = R^- \\ X_4 + X_{15} = S^+; &\qquad X_4 - X_{15} = S^- \end{aligned} \tag{22}$$

These are of the form (17). Therefore the eigenfunctions of X_5 are P^+, Q^+, R^+, S^+, each having an eigenvalue (root) $+1$, and P^-, Q^-, R^-, S^-, each having an eigenvalue (root) -1. We put these in sets:

$$L^{(+1)} = \{P^+, Q^+, R^+, S^+\} \tag{23a}$$

the root $\alpha = +1$ having multiplicity 4; and

$$L^{(-1)} = \{P^-, Q^-, R^-, S^-\} \tag{23b}$$

the root $-\alpha = -1$ having multiplicity 4. The roots of $so(5, 1)$ are therefore given as

$$\begin{array}{ll} \alpha = +1, & -\alpha = -1 \\ \beta = +1, & -\beta = -1 \\ \gamma = +1, & -\gamma = -1 \\ \delta = +1, & -\delta = -1 \end{array} \tag{24}$$

The null root has multiplicity 7. The roots are all normalized. If we denote by Δ the set of all roots of $so(5, 1)$, we have

$$so(5, 1) = L^0 \oplus \sum_{\alpha \varepsilon \Delta} L^\alpha = L^0 + L^{(+1)} \oplus L^{(-1)} \tag{25}$$

2.3. Hyperplanes and Weyl Chambers of the Group of $so(5, 1)$

The hyperplanes associated with the roots $\alpha = +1$, $\beta = +1$, $\gamma = +1$, $\delta = +1$ are simply a null point, which is the origin. These are

$$P_\alpha \equiv P_{+1} = 0; \qquad P_\beta \equiv P_{+1} = 0; \qquad P_\gamma \equiv P_{+1} = 0; \qquad P_8 \equiv P_{+1} = 0$$

We next calculate the Weyl reflection operator S'_α for $so(5, 1)$ associated with the hyperplane α, where S_α has the following properties:

(i) $S_\alpha \mathbf{H} = \mathbf{H} - \alpha(H)\mathbf{H}_\alpha = \mathbf{H} - B(\mathbf{H}, \mathbf{H}'_\alpha)\mathbf{H}_\alpha$ $(\mathbf{H} \in L_A)$.
(ii) $S_\alpha P_\alpha = P_\alpha$ for any hyperplane P_α.
(iii) $S_\alpha^2 = I$.
(iv) $B(\mathbf{H}, \mathbf{H}) = B(S_\alpha \mathbf{H}, S_\alpha \mathbf{H})$.
(v) $B(\mathbf{H}, \mathbf{H}_\alpha) = 0$, for any $\mathbf{H}$ lying wholly on the hyperplane P_α.
(vi) $S_\alpha \mathbf{H}_\alpha = -\mathbf{H}_\alpha$.

Since L_A is one-dimensional, for $so(5, 1)$ one obtains simply that

$$X_5 = H \in L_A$$

Therefore

$$S_+ X_5 = X_5 - 2X_5 = X_5; \qquad (S_{+1})^2 X_5 = S_+(-X_5) = +X_5; \qquad \text{etc.} \tag{26}$$

Therefore the Weyl reflection operators consist of S_+, S_-, and I. The Weyl chambers for $so(5, 1)$ are the spaces between any two hyperplanes of $so(5, 1)$ in L_A. These consists of the line segments

$$C_+ = (0, +\infty) \qquad \text{and} \qquad C_- = (0, -\infty) \tag{27}$$

2.4. The Nilpoint Subalgebra of $so(5, 1)$

From the commutation relations

$$\begin{aligned} &[X_5, P^+] = P^+; \quad &[X_5, Q^+] = Q^+ \\ &[X_5, R^+] = R^+; \quad &[X_5, S^+] = S^+ \\ &[X_5, P^-] = -P^-; \quad &[X_5, Q^-] = -Q^- \\ &[X_5, R^-] = -R; \quad &[X_5, S^-] = -S^- \end{aligned} \tag{28}$$

one obtains that the eigenfunctions of the positive and negative roots are, respectively, P^+, Q^+, R^+, and S^+ and P^-, Q^-, R^-, and S^-. A standard theorem (Ndili *et al.*, 1975) leads us to the prescription that these positive root eigenfunctions form a nilpotent Abelian subalgebra of the parent Lie algebra $so(5, 1)$. This Lie subalgebra is denoted by $L_N^+ = \{P^+, Q^+, R^+, S^+\}$. Similarly, P^-, Q^-, R^-, and S^- form a nilpotent Abelian subalgebra L_N^-.

3. THE IWASAWA FACTORIZATION OF $so(5, 1)$ AND $SO(5, 1)$

Having obtained the above detailed information about the structure of the Lie algebra $so(5, 1)$, one can complete the problem of decomposition of the algebra by appealing to the well-known theorem (Hermann, 1966; Helgason, 1962) according to which a Lie algebra L can uniquely be written in the form

$$L = L_K \oplus L_A \oplus L^{+)N} \tag{29}$$

For $SO(5, 1)$ this means

$$so(5, 1) = so(5, R) \oplus X_5 \oplus L_N^+, \qquad L_N^+ = \{P^+, Q^+, R^+, S^+\} \tag{30}$$

At the Lie group level, we have the Iwasawa factorization of the connected analytic group $G \equiv SO(5, 1)$, whose corresponding Lie algebra is $L \equiv so(5, 1)$. In general, if K, A, and N^+ stand for the connected analytic subgroups of G that correspond to the Lie subalgebras L_K, L_A, and L_N^+, respectively, the Iwasawa factorization of G is

$$G = K \cdot A \cdot N^+ \tag{31}$$

where A is the Abelian subgroup of G, N^+ is the nilpotent Abelian subgroup of G, and K is the invariant compact subgroup of G.

These groups can be parametrized in the usual way. We first construct suitable matrix representations of the generators of $SO(5,1)$ and later parametrize the subgroups A, K, and N^+. For the matrix representations for the generators of $SO(5,1)$, we let A be the matrix generator of the pseudo-orthogonal group $SO(n-s,s)$. The condition to be satisfied by A is

$$A^T g A = g \tag{32}$$

where g is the metric connection in the space. We rewrite (32) as

$$A^T = gA^{-1}g \tag{33}$$

and put $A = e^X$, where X is an arbitrary, $n \times n$ matrix. Then

$$A^T = gA^{-1}g$$

becomes

$$e^{X^T} = ge^{-X}g^{-1} = ge^{-X}g^{-1} = e^{-gXg-1}$$

or $X^T = -gXg^{-1}$, giving

$$X^T g + gX = 0. \tag{34}$$

If the generators of $SO(n-s,s)$ are Z_{ij}, we have

$$Z_{ij}^T g + Z_{ij} = 0 \tag{35}$$

where

$$g = \left[\begin{array}{c|c} I_{n-s} & 0 \\ \hline 0 & -I_s \end{array}\right]$$

with I_{n-s} as an identity $(n-s)\times(n-s)$ matrix, while I_s is an identity $s \times s$ matrix. If

$$Z_{ij} = \left[\begin{array}{c|c} Z_1 & Z_2 \\ \hline Z_3 & Z_4 \end{array}\right]$$

then $Z_{ij}^T g + gZ_{ij} = 0$ gives

$$\left[\begin{array}{c|c} Z_1^T + Z_1 & Z_2 - Z_3 \\ \hline Z_2^T - Z_3 & -(Z_4^T + Z_4) \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right] \tag{36}$$

giving:

(i) $Z_1^T + Z_1 = 0$, or $Z_1^T = -Z_1$, so that Z_1 is an antisymmetric matrix of order $(n-s)\times(n-s)$.

(ii) $Z_2 - Z_3^T = 0$, or $Z_3^T = Z_2$, so that Z_2 is an $(n-s)\times s$ matrix.

(iii) $Z_4^T = -Z_4$, so that Z_4 is an antisymmetric matrix of order $s \times s$.

Therefore

$$Z_{ij} = \left[\begin{array}{c|c} Z_1 & Z_2 \\ \hline Z_2^T & Z_4 \end{array}\right] \tag{37}$$

For the $SO(5, 1)$ we have that Z_1 is a 1×1 null matrix, Z_2 is a null 3×1 matrix, and $Z_3 = Z_2^T$. In general, we can always choose for the generators of $SO(n-s, s) \equiv SO(p, q)$ the matrix representation

$$\begin{aligned} Z_{ij} &= e_{ij} - e_{ji}, && i, j \leq p \\ &= -e_{ij} + e_{ji}, && i, j > p \\ &= e_{ji} + e_{ji}, && i \leq p, \quad j > p \\ &= -e_{ij} - e_{ji}, && i > p, \quad j \leq p \end{aligned} \tag{38}$$

where e_{ij} is the matrix with 1 at the ith row and the jth column, but otherwise zero everywhere. In the case of $SO(5, 1)$, $p = 5$, and so Z_{12}, Z_{13}, Z_{14}, Z_{15}, Z_{23}, Z_{24}, Z_{25}, Z_{34}, Z_{35}, Z_{45}, Z_{16}, Z_{26}, Z_{36}, Z_{46}, and Z_{56} are the generators. Following the identifications in (3), we calculate explicitly the following:

$$X_1 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad X_2 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad X_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad X_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X_8 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{39}$$

$$X_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_9 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad X_{14} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$X_{15} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

3.1. Parametrization of the Abelian Subgroup A

If an arbitrary element of A is a, then

$$a=\exp[\theta X_5]=I+\theta X_5+\theta^2 X_5^2/2!+\cdots \tag{40}$$

where θ is an arbitrary real parameter of the Lie group A, with $-\infty\leq\theta\leq+\infty$. We substitute the matrix representation of X_5 in (40) to get

$$a=\begin{bmatrix} \cosh\theta & 0 & 0 & 0 & 0 & \sinh\theta \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \sinh\theta & 0 & 0 & 0 & 0 & \cosh\theta \end{bmatrix} \tag{41}$$

3.2. The Maximal Compact Subgroup K

If an arbitrary element of K is k, then

$$k=e^{\theta_1 X_1}e^{\theta_2 X_2}e^{\theta_3 X_3}e^{\theta_4 X_4}e^{\theta_6 X_6}e^{\theta_7 X_7}e^{\theta_8 X_8}e^{\theta_{10} X_{10}}e^{\theta_{11} X_{11}}e^{\theta_{13} X_{13}} \tag{42}$$

where θ_i $(i=1,2,3,4,6,7,8,10,11,13)$ are real, bounded parameters. We obtain the following parametrized representations:

$$e^{\theta_1 X_1}=\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{43}$$

$$e^{\theta_2 X_2}=\begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{44}$$

$$e^{\theta_3 X_3} = \begin{bmatrix} \cos\theta_3 & 0 & 0 & -\sin\theta_3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \sin\theta_3 & 0 & 0 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{45}$$

$$e^{\theta_4 X_4} = \begin{bmatrix} \cos\theta_4 & 0 & 0 & 0 & -\sin\theta_4 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \sin\theta_4 & 0 & 0 & 0 & \cos\theta_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{46}$$

$$e^{\theta_6 X_6} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta_6 & -\sin\theta_6 & 0 & 0 & 0 \\ 0 & \sin\theta_6 & \cos\theta_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{47}$$

$$e^{\theta_7 X_7} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta_7 & 0 & -\sin\theta_7 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \sin\theta_7 & 0 & \cos\theta_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{48}$$

$$e^{\theta_8 X_8} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta_8 & 0 & 0 & -\sin\theta_8 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \sin\theta_8 & 0 & 0 & \cos\theta_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{49}$$

$$e^{\theta_{10}X_{10}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta_{10} & -\sin\theta_{10} & 0 & 0 \\ 0 & 0 & \sin\theta_{10} & \cos\theta_{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{50}$$

$$e^{\theta_{11}X_{11}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta_{11} & 0 & -\sin\theta_{11} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \sin\theta_{11} & 0 & \cos\theta_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{51}$$

$$e^{\theta_{13}X_{13}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_{13} & -\sin\theta_{13} & 0 \\ 0 & 0 & 0 & \sin\theta_{13} & \cos\theta_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{52}$$

One finally multiplies out the parametrized matrix representation (43)-(52) to obtain the parametrized representation of an arbitrary element kaK of the maximal compact subgroup K of $SO(5, 1)$.

3.3. The Parametrization of N^+ of $SO(5, 1)$

For any arbitrary element n^+ of N^+, we have

$$n^+ = e^{d_1P^+} e^{d_2Q^+} e^{d_3R^+} e^{d_4S^+} \tag{53}$$

where the d_i $(i = 1, 2, 3, 4)$ are real parameters, with $-\infty \le d_i \le +\infty$. Now,

$$P^+ = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{54}$$

$$Q^{+}=\begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{55}$$

$$R^{+}=\begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{56}$$

$$S^{+}=\begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{57}$$

$$P^{+^2}=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -1 \\ & & \bigcirc & & & \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{58}$$

P^{+^3} = a 6×6 null matrix

$$Q^{+^2}=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -1 \\ & & \bigcirc & & & \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{59}$$

Q^{+^3} = a 6×6 null matrix

$$R^{+^2}=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -1 \\ & & \bigcirc & & & \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{60}$$

R^{+^3} = a 6×6 null matrix

Finally,

$$S^{+2} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -1 \\ & & \bigcirc & & & \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{61}$$

$S^{+3} = \text{a } 6\times 6 \text{ null matrix}$

$$e^{d_1 P^+} = \begin{bmatrix} 1-\frac{1}{2}d_1^2 & -d_1 & 0 & 0 & 0 & -\frac{1}{2}d_1^2 \\ d_1 & 1 & 0 & 0 & 0 & d_1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{2}d_1^2 & d_1 & 0 & 0 & 0 & 1+\frac{1}{2}d_1^2 \end{bmatrix} \tag{62}$$

$$e^{d_2 Q^+} = \begin{bmatrix} 1-\frac{1}{2}d_2^2 & 0 & -d_2 & 0 & 0 & -\frac{1}{2}d_2^2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ d_2 & 0 & 1 & 0 & 0 & d_2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{2}d_2^2 & 0 & d_2 & 0 & 0 & 1+\frac{1}{2}d_2^2 \end{bmatrix} \tag{63}$$

$$e^{d_3 R^+} = \begin{bmatrix} 1-\frac{1}{2}d_3^2 & 0 & 0 & -d_3 & 0 & -\frac{1}{2}d_3^2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ d_3 & 0 & 0 & 1 & 0 & d_3 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{2}d_3^2 & 0 & 0 & d_3 & 0 & 1+\frac{1}{2}d_3^2 \end{bmatrix} \tag{64}$$

$$e^{d_4 S^+} = \begin{bmatrix} 1-\frac{1}{2}d_4^2 & 0 & 0 & 0 & -d_4 & -\frac{1}{2}d_4^2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ d_4 & 0 & 0 & 0 & 1 & d_4 \\ \frac{1}{2}d_4^2 & 0 & 0 & 0 & d_4 & 1+\frac{1}{2}d_4^2 \end{bmatrix} \tag{65}$$

Substituting equations (62)-(65) in (53), one obtains the parametrized matrix representation of an arbitrary element n^+ of the nilpotent Abelian subgroup N^+ of $SO(5, 1)$.

Finally, we obtain an arbitrary element g of $SO(5, 1)$ in the parametrized Iwasawa form:

$$g = akn^{+} \tag{66}$$

by substituting from (41), (43)-(52), and (62) into (65).

REFERENCES

Gourdin, M. (1967). *Unitary Symmetry*, North-Holland, Amsterdam.

Helgason, S. (1962). *Differential Geometry and Symmetric Spaces*, Advanced Academic Press, New York.

Hermann, R. (1966). *Lie Groups for Physicists*, Benjamin, New York.

Nagel, B. (1969). Lie Algebras, Lie Groups, Group Representations and Some Applications to Problems in *Elementary Particle Physics*, Seminar Notes, Department of Theoretical Physics, Royal Institute of Technology, Stockholm.

Ndili, F. N., Nagel, B. (1969). Chukwumah, G. C., and Okeke, P. N. (1975). Decomposition theorems and structural parameters of classical Lie algebras (unpublished).

Pontryagin, L. S. (1969). Topological group representations and some applications to problems of elementary particle physics, Seminar Notes, Department of Theoretical Physics, Royal Institute of Technology, Stockholm.

Strom, S. (1971). Introduction to the theory of groups and group representations, Lecture Notes, CPT 120, Centre for Particle Theory, Austin, Texas.